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Shadows Produced by Spin-Stabilized Communication Satellites

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A spin-stabilized satellite generally utilizes a so-called toroidal antenna that has a null cone suı rounding the axis of symmetry (spin axis) The gıeatest gain over isotıopic antennas occurs with the widest possible null cone However, wide null cones may cause shadows on the earth and prohibit communication between mutually visible ground stations and the satellite It is shown how the size and shape of the shadowed zones may be calculated for arbitrary orientations of the spin axis It is also shown how to find the narrowest possible antenna beam, which avoids any null-cone shadow on earth when the spin axis is maintained perpendicular to the ecliptic plane (desirable for thermal reasons) For near polar orbits, it is not possible to escape null-cone shadow and still realize any appreciable antenna gain if the spin axis is maintained exactly perpendicular to the ecliptic plane However, it is shown that a ıeasonably small tilt of the spin axis will either reduce the shadow to an acceptable level or eliminate it entirely The critical tilt angles, for complete elimination of null-cone shadow, and the effect of nodal regression due to earth oblateness may be found from given curves

Nomenclature

A_p = projected area seen from satellite
A_{p0} = projected area of effective earth disk
A_p' = projected area of spherical cap (see Fig 5)
A = surface area of null-cone shadow
$A_{ 0}$ = surface area of effective visible earth
$A '$ = surface area of spherical cap (see Fig 5)
e_m = minimum elevation angle of antenna on earth
H = geocentric altitude of satellite
i = inclination of orbit referred to equator
i' = inclination or orbit referred to ecliptic
I_p = $A_p - A_p'$
I_s = $A - A_s'$
R = radius of earth
u = argument of latitude referred to equator
u' = argument of latitude referred to ecliptic
α_1 = angle subtended, at satellite, by effective earth limb
β = acute angle between spin axis and satellite earth line
β_1 = $\alpha_1 + \gamma$
β_m = minimum value of β anywhere in a given orbit
β_{0m} = value of β_m when spin axis is perpendicular to ecliptic
γ = semivertex angle of null cone
γ_0 = fixed null-cone angle
$\gamma_{m x}$ = $\beta_m - \alpha_1$
γ^* = $90 - \alpha_1$
ϵ = obliquity of ecliptic (taken as $23° 27'$)
θ = spherical coordinate (colatitude) of point where cone generator pierces earth surface
θ_1 = θ evaluated at σ_1
θ_2 = $90° - e_m - \alpha_1$
ν = $u - u'$
σ = angle between satellite earth line and generic cone generator
$\sigma_1 \sigma_2$ = limit ing values of σ
τ = tilt angle of spin axis
τ = $\alpha_1 + \gamma_0 - \beta_{0m}$
Φ = spherical coordinate (longitude) of point where cone generator pierces earth surface
Ω = longitude of node measured along equator
Ω' = longitude of node measured along ecliptic

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1 Introduction

A NUMBER of papers[1–4] have been published on problems associated with earth coverage by communication satellites These have, for the most part, been concerned with the probability of mutual visibility between selected

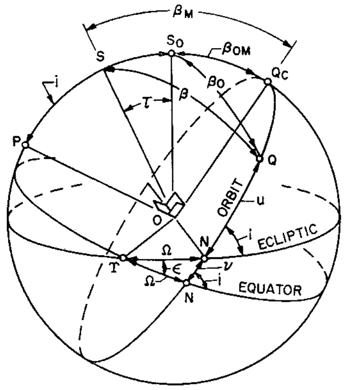

Fig 1 Celestial sphere showing orbital parameters

earth stations, with the choice of orbital parameters and number of satellites used, and the most favorable methods of synchronizing the motions of various satellites These broad system studies have all been based on the assumption tion that line-of-sight visibility is sufficient for communication between satellite and earth station Such an assumption would be valid for satellites with earth-pointing antennas, with omnidirectional antennas, or with a suitable attitude control system for properly pointing the satellite's antenna

For the case of a spin-stabilized satellite whose antenna pattern is symmetric about the spin axis, line-of-sight visibility does not necessarily insure communication between satellite and ground station In general, the antenna will have dead zones or null cones surrounding the spin axis † If a particular ground station is shadowed by this null cone, no communication is possible between ground station and satellite even though the satellite is on a line of sight

To the best of this author's knowledge, no studies have been published on system performance which take into account the effect of null-cone shadows It is the purpose of this paper to show how the extent of the null-cone shadow may be calculated for any given spin axis orientation and any given satellite position Although it is not the purpose of this paper to present a systems performance study that takes into account the important question of mutual visibility between ground stations and satellites, it may readily be seen that knowledge of null-cone shadows is an important prerequiste for such a study Furthermore, one direct measure of system performance, namely, the ratio of earth surface actually illuminated by a given satellite to that illuminated by the same satellite with an omnidirectional antenna, is presented herein

The extent of null-cone shadow has important implications for antenna design, since it influences the width of the antenna beam and thereby affects the extremely important parameter of antenna gain It will also be shown below that it is possible for a wide variety of orbits to use directional antennas and still avoid any null-cone shadow on the earth, thereby affording appreciable antenna gain over omnidirectional (isotropic) antennas In particular, if the spin axis is kept perpendicualr‡ to the ecliptic plane, it will be shown how the null-cone angle (and hence antenna gain) depends explicitly upon orbital inclination and longitude if the null cones are required to clear the earth's surface It will be seen below that, for a fixed antenna angle, a certain amount of null-cone shadow is inescapable if the spin axis is kept perpendicular to the ecliptic plane, but if the spin axis is tilted in a certain well-defined manner, the shadow may be reduced or even eliminated

† For some applications the spin axis is surrounded by a horn antenna Most of the geometric results to be presented apply equally well for coverage or lack of coverage by horn antennas, with only minor reinterpretations

‡ This orientation is desirable for thermal reasons and was, in fact, essentially achieved in the TELSTAR satellite

In Sec 2, the conventions used for describing the satellite's motion in orbit are defined, and in the following section, criteria for the presence or absence of null-cone shadow are established for arbitrary spin-axis orientation In Sec 4, the method used to find the minimum angle β_{0m} between the geocentric radius vector to the satellite and the normal to the ecliptic plane is shown The angle β_{0m} is an important parameter, which enables one to compute a number of other important quantities such as the critical tilt angle τ required to clear the earth completely of null-cone shadow; this calculation is made in Sec 5 In the next section, the shape and size of the null-cone shadow on earth are calculated for arbitrary satellite positions The manner in which nodal regression (caused by earth oblateness) influences the null-cone shadow is described briefly in Sec 7 Finally, conclusions are stated in Sec 8

2 Orbit Description

For simplicity of exposition, the satellite will be assumed to move along a great circle of the celestial sphere centered at the geocenter designated by O in Fig 1 The ecliptic plane intercepts this sphere in a great circle whose "north" pole is designated by S_0 in Fig 1 The equatorial plane of the earth intersects the ecliptic plane in a directed line which points from the earth to the sun at the time of the vernal equinox The point at which this line intercepts the celestial sphere will be referred to as the vernal equinox and designated by ϒ The acute dihedral angle between the ecliptic and equatorial planes will be designated by ϵ (obliquity of ecliptic, $\epsilon \doteq 23° 27'$)

The satellite's orbit intersects the earth's equator in the node N, as shown in Fig 1 The orientation of the orbital plane is fully specified by the "longitude of the node" Ω and the "inclination" i, where

$$0 \leq i \leq 180° \qquad (2\,1)$$

$$0 \leq \Omega \leq 180° \qquad (2\,2)$$

The angle Ω is measured eastward, along the equator, from the vernal equinox to the node N, and i is the dihedral angle between the orbital plane and the equatorial plane For future reference, it is noted that the perpendicular to the orbit plane pierces the celestial sphere at a point P in the northern hemisphere if $0 < i < 90°$; P is in the southern hemisphere if $90° < i < 180°$ Although the sense of motion along the orbit is not of primary interest, it may be assumed without loss of generality that the motion is direct,§ so that N is the "ascending" node (where the satellite crosses the equator from south to north) if $0 < i < 90°$, and N is the "descending" node if $90° < i < 180°$ For retrograde orbits, the analysis is still applicable if we interchange "ascending" and "descending" in the preceding sentence An arbitrary position of the satellite in its orbit plane, such as Q in Fig 1, is defined by the angle u (equal to arc NQ) subtended at the geocenter by the satellite and the node N; u is sometimes referred to as the "argument of the latitude "

The elements of the orbit are sometimes conveniently defined with respect to the ecliptic rather than the equator; for example, the longitude of the node Ω' and the inclination i' referred to ecliptic coordinates are measured as shown in Fig 1, and the argument of the latitude referred to the ecliptic is arc $N'Q$, which will be referred to as u' The relationship between u and u' is

$$u' = u - \nu \qquad (2\,3)$$

where $\nu = \text{arc}NN'$ shown in Fig 1, and where N' represents the intersection of the orbit and the ecliptic

§ If the satellite appears when viewed from the North Pole to move about the pole counterclockwise, the motion is said to be 'direct'; otherwise the motion is said to be 'retrograde '

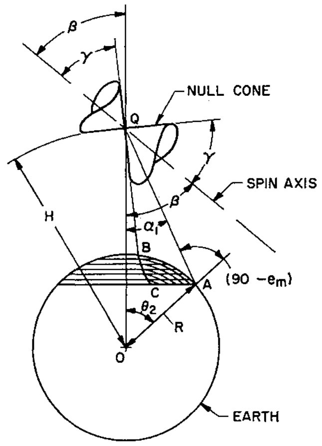

Fig 2a View of earth and null cone showing null-cone shadow *ABC*

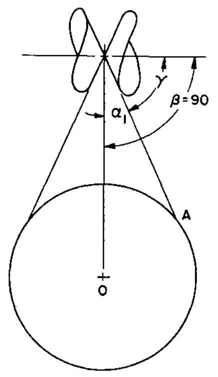

Fig 2b Null-cone axis shown perpendicular to orbital plane

3 Interception of Null Cone with Earth

The satellites under investigation are assumed to have antenna patterns that are symmetrical about the spin axis In particular, it is assumed that the antenna has a so-called toroidal pattern, that is, the spin axis is surrounded by both nappes of a cone within which no energy is radiated (see Fig 2a) In other terms, the satellite cannot communicate with any part of the earth's surface which lies inside one of these so-called "null cones"; such a surface element will be referred to as "in the null-cone shadow " Figure 2a shows a view looking in on the plane formed by the spin axis of the satellite and the radius vector OQ from the geocenter to the satellite at points Q; the geocentric altitude OQ will be referred to as H If the transmitting antenna at Q were isotropic, and if the minimum elevation angle of a receiving antenna on the earth were e_m, the satellite could communicate with all points on the ' upper" hemisphere of the earth within the cone of semivertex angle $AOQ = \theta_2$ The point A is uniquely fixed by the requirement that $\sphericalangle QAO = 90° + e_m$ The surface area on the earth, which "sees" the isotropic antenna at Q, will be referred to as the effective visible cap and is shown with horizontal hatching in Fig 2a The surface area of this cap A_0 is given by

$$A_0 = 2\pi r^2(1 - \cos\theta_2) \qquad (3\ 1)$$

The area of the cap projected onto the diametral plane, perpendicular to OQ, is designated by A_{p0} and given by

$$A_{p0} = \pi R^2 \sin^2\theta_2 \qquad (3\ 2)$$

The angle θ_2 is easily expressed in terms of the angle $\alpha_1 = \sphericalangle AQO$, since

$$\theta_2 = 90° - e_m - \alpha \qquad (3\ 3)$$

and

$$\sin\alpha_1 = (R/H)\cos e_m \qquad (0 \leq \alpha_1 \leq 90°) \qquad (3\ 4)$$

The null-cone shadow will be bounded by the curvilinear triangle ABC on the surface of the sphere and a similar triangle on the other side of arc BA The surface area A_S of the null-cone shadow is shown doubly hatched in Fig 2a, and the projection of this area on a plane perpendicular to the line OQ will be designated by A_p Both of these areas will be calculated in Sec 6

It is apparent that, for a given altitude, the size of the null-cone shadow on the earth depends only upon the semivertex angle of the null cone γ and the acute angle β $(0 \leq \beta \leq 90°)$ between the spin axis and the radius vector It is seen in Fig 2a that the null-cone shadow exists only when β is less than a critical angle β_1, defined by $\beta_1 - \gamma = \alpha_1$, i e ,

$$\beta_1 = \alpha_1 + \gamma \qquad (3\ 5)$$

For a given null cone, the extent of the shadowed area may be decreased by rotating the spin axis so as to increase β, and for a given β, the shadow may be decreased by decreasing γ, thereby narrowing the null cone

Both of these alternatives will be pursued further, but befoıe examining the problem quantitatively, some geneıal design criteria will be considered:

1) It is desirable to eliminate, if practicable, the null-cone shadow in order to insuıe the widest possible earth coverage

2) It is desirable to use as wide a null cone as possible in order to concentrate the antenna beam and provide high gain

3) It is desirable from the point of view of thermal efficiency, solar cell power output, and uniformity of service to have the spin axis perpendicular to the ecliptic plane

Because of criterion 1, attention will now be focused on systems which completely eliminate the null-cone shadow, although later on (see Sec 6), the effect of relaxing this constraint will be examined Thus, until further notice, it will be assumed that

$$\beta > \beta_1 = \alpha_1 + \gamma \qquad (3\ 6)$$

One important limiting case is to have β achieve its maximum value of 90° as shown in Fig 2b Under this circumstance, the null-cone angle γ may achieve its maximum value γ^* defined by

$$\gamma^* = 90° - \alpha_1 \qquad (3\ 7)$$

in accordance with criterion 2 If this highest gain condition is to persist at all times, β must remain a right angle at all times, requiring the spin axis to remain perpendicular to the orbital plane at all times Some aspects of this type of orientation have been examined in detail in Ref 5 In brief summary, some of the disadvantages of this system which must be weighed against its advantage of high antenna gain, are as follows:

1) Because of nodal ıegression due to earth oblateness and other perturbing effects, the oıbital plane precesses with time, whereas the spin axis tends to remain fixed in inertial space This causes a decrease in angle β which makes the null cone shadow the earth, unless a suitable tolerance is placed on the cone angle γ, thereby decreasing its gain (Although this statement is equally true if the spin axis is maintained perpendicular to the ecliptic plane, the effects of nodal regression are far less pronounced in this latter case; see Sec 7) For nonpolar orbits above a critical inclination, which depends on longitude of node, it is not possible to avoid completely null-cone shadows for long periods of time unless one uses isotropic antennas

2) No attempt is made in this scheme to satisfy criterion 3, and the solar plant must be relatively isotropic in design, since it sees the sun from all angles during the course of the year Furthermore, the sun rays may be parallel to the spin axis, causing some solar cells to run at a relatively high (and inefficient) temperatuıe

3) When several satellites are launched from the same rocket vehicle, it is necessary to impart to each of them some relative velocity in the direction of the orbital motion in order that the satellites separate and do not bunch together throughout their useful lifetimes If the spin axis is perpendicular to the orbital plane, it is necessary to impart relative velocities to each satellite perpendicular to its spin axis Doing this reliably, without inducing wobbling (or precession) of the spin axis, may be a technical problem of some magnitude On the other hand, it should be relatively easy to eject satellites accurately along their spin axes with reasonable velocity differences If the spin axis is not

perpendicular to the orbit plane, any impulse along the axis will, in general, have a finite component in the direction of the trajectory; therefore, properly timed velocity increments along the spin axis will lead to separation of the satellites if the spin axis is not kept perpendicular to the orbit plane

It is worth noting that, if the orbital plane happens to be the plane of the ecliptic, criteria 1, 2, and 3 may all be satisfied simultaneously, and we have an ideal, high-gain system with good thermal properties From a communications point of view, it is necessary to have satellites in other orbits as well (particularly, in near polar orbits); therefore, more general orbits will be examined below

Henceforth in this paper it will not be specifically required that the spin axis be perpendicular to the orbital plane; in other terms, criterion 2 will be relaxed, and simultaneous satisfaction of criteria 1 and 3 will be sought It may be observed from Fig 2b that, if $\gamma < 90° - \alpha_1$, only one of the two nappes of the null cone can intersect the effective earth at a given time (as shown in Fig 2a) regardless of the spin axis orientation It will be assumed henceforth that $\gamma < 90° - \alpha_1$, so that Fig 2a remains valid

If the spin axis is not perpendicular to the orbital plane, the angle β will not be constant throughout the orbit, but will vary between some maximum value and a minimum value β_m If one insists upon enforcement of criterion 3 (spin axis perpendicular to ecliptic plane), β_m will depend only upon the orbital parameters Ω and i If one insists upon enforcement of criterion 1 (no null-cone shadow), it is necessary that

$$\beta_m \geq \beta_1 = \alpha_1 + \gamma \qquad (3\ 8)$$

Of course, inequality (3 8) may be satisfied only if

$$\beta_m \geq \alpha_1 \qquad (3\ 9)$$

Finally, if one wishes to maximize the antenna gain (criterion 2), it is necessary to choose the maximum possible value of γ consistent with inequality (3 8), that is,

$$\gamma_{\max} = \beta_m - \alpha_1 \qquad (3\ 10)$$

We thus see that the choice of the highest possible gain toroidal antenna, which never shadows the earth, and whose symmetry axis remains perpendicular to the ecliptic, depends only upon the critical angle β_m, which we now proceed to find

4 Location and Magnitude of β_m

The acute angle β_0 between the spin axis direction OS_0 (perpendicular to ecliptic plane) and an arbitrary radius vector OQ is nothing more than the celestial colatitude (referred to ecliptic coordinates) and is clearly a minimum at point Q (see Fig 1), where the satellite rises highest in the hemisphere above the ecliptic plane The point Q_c is clearly located where the arc $N'Q = 90°$ In other words, the directed¶ arc $Q\,S_0$ is just the complement of the orbit's inclination with respect to the ecliptic

$$\text{arc}Q\,S_0 = 90° - i' \qquad (4\ 1)$$

Since β_0 is defined as the acute angle between spin axis and radius vector without reference to sign (see Fig 2a), it follows that

$$\min\beta_0 = \beta_{0m} = |90° - i'| \qquad (0 < \beta_{0m} < 90°) \quad (4\ 2)$$

Equation (4 2) can also be written in the equivalent form

$$\sin\beta_{0m} = |\cos i'| \qquad (4\ 3)$$

Since it is conventional to specify the orbits of earth satellites with respect to the equator, we shall express i' in terms of the parameters i and Ω measured with respect to the equator as shown in Fig 1 Similarly, it will prove convenient to express the longitude Ω' of the node N' (referred to ecliptic coordinates) and the angle between the two nodes $\nu = NN'$ in terms of the known quantities i, Ω, ϵ This is most simply done by applying the spherical law of cosines in polar form** three times in succession (with cyclic permutation of angles and sides) to the spherical triangle ♈$N'N$, yielding

$$\cos i' = \cos\epsilon \cos i + \sin\epsilon \sin i \cos\Omega \qquad (4\ 4)$$

$$\cos\Omega' = (\cos\epsilon \cos i' - \cos i)/\sin\epsilon \sin i' \qquad (4\ 5)$$

$$\cos\nu = (\cos\epsilon - \cos i \cos i')/\sin i \sin i' \qquad (4\ 6)$$

The angles i', Ω', and ν are completely defined by the preceding expressions in their range of definition between 0° and 180° The special case where $i = 0$ refers to an equatorial orbit where the point N is not uniquely defined However, if we agree always to associate $\Omega = 0$ with $i = 0$, the previous formulas will, in the limit, predict that $i' = \epsilon$, $\Omega' = \nu = 180°$ for equatorial orbits

The special case where $i' = 0$ refers to an ecliptic orbit for which ν is undefined and never actually of interest We now see from Eqs (4 2) and (4 3) that the acute angle β_{0m} is given by $\beta_{0m} = |90° - i'|$, or

$$\beta_{0m} = \sin^{-1}|\cos\epsilon \cos i + \sin\epsilon \sin i \cos\Omega| \qquad (4\ 7)$$

In short, we see that if $\beta_{0m} > \alpha_1$ [inequality (3 9)] it is possible to satisfy inequality (3 8) with a finite null cone whose semivertex angle γ is given by Eqs (3 10) and (4 7) Figure 3a shows β_{0m} in terms of orbital elements Ω and i

If, for any combination of i and Ω, $\beta_{0m} < \alpha_1$ (as it necessarily must be for any finite values of α_1 and some values of i), γ must be zero to avoid null-cone shadow That is, no antenna gain is possible in these cases, as long as one insists upon keeping the spin axis perpendicular to the ecliptic

If, for example, a satellite is at an altitude of 6000 naut miles (9436 naut miles geocentric) and the ground antennas have a minimum elevation angle of $e_m = 7\frac{1}{2}°$, Eq (3 4) shows that $\alpha_1 = 21\ 15°$ From Fig 3a, it is a simple matter to find the maximum gain antenna for any given orbit via Eq (3 10) in the form $\gamma_{\max} = \beta_{0m} - \alpha_1$ Figure 3b shows the antenna angle $\gamma_{m\ x}$ corresponding to various circular orbits at a 6000-naut-mile altitude such that the earth is never shadowed and the spin axis remains perpendicular to the ecliptic

In order to realize some gain for those orbits where $\beta_{0m} < \alpha_1$ (especially the very important near polar orbits), one may elect to relax criterion 3, and not keep the spin axis exactly normal to the ecliptic plane Another reason for relaxing criterion 3 is that it is not always possible to design an antenna with very narrow null cones; hence, one may be forced to utilize an antenna with a fixed minimum γ (called γ_0), which exceeds the maximum value predicted by (3 10), thereby making the preceding scheme valueless

5 Tilting Spin Axis off Ecliptic

In this section, it will be assumed that the antenna has a fixed null cone of semivertex angle γ_0, and a spin axis orientation will be sought which keeps the earth out of the null-cone shadow completely (criterion 1, enforced) but at the same time does not depart too much from normality to the ecliptic plane (criterion 3, relaxed but not abandoned) This compromise is readily achieved by tilting the spin axis off its initial direction OS_0 by an angle τ about the nodal axis ON' as shown in Fig 1, where OS represents the direction of the tilted spin axis It is clear from Fig 1 that, as the satellite

¶ The arc is measured positively from Q to S_0 in the sense of rotation of a right-handed screw advancing along the direction of ON'

** If A, B, C and a, b, c represent the angles and sides of a spherical triangle, the "polar form" of the law of cosines is $\cos C = -\cos A \cos B + \sin A \sin B \cos c$; see Ref 6, for example

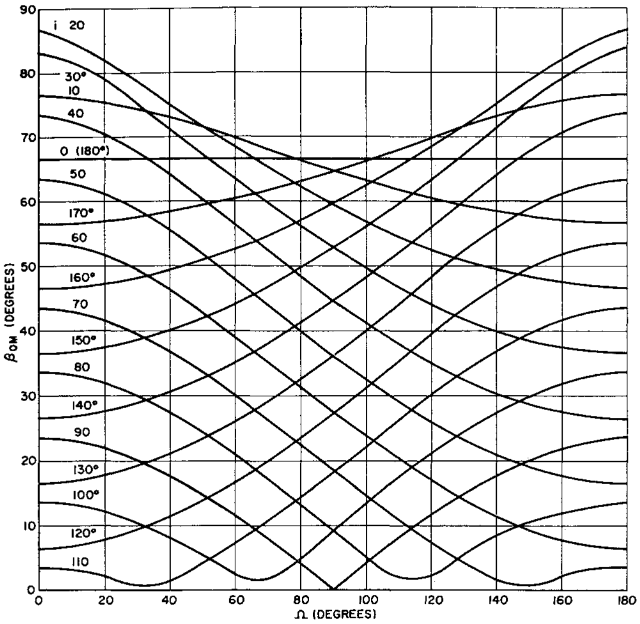

Fig 3a Minimum acute angle β_{0m} between satellite radius vector and perpendicular to ecliptic for various circular orbits

moves in orbit, the angle $\beta = SQ$ achieves its minimum value when the satellite is in its critical position Q_c (where $u' = 90°$) Therefore, it follows from inequality (3 6) or Fig 2a that, if the minimum value of β defined by $SQ = \beta_m$ exceeds $\alpha_1 + \gamma$, the null-cone shadow never intercepts the effective disk of the earth In other terms, there will never be any shadow if

$$\beta_m = \beta_{0n} + \tau \geq \alpha_1 + \gamma_0 \qquad (5\ 1)$$

or

$$\tau \geq \tau_c = \alpha_1 + \gamma_0 - \beta_{0m} \qquad (5\ 2)$$

If one wished to keep the spin axis as close as possible to perpendicularity with the ecliptic, it is necessary only to rotate through the angle $\tau_c = \alpha_1 + \gamma_0 - \beta_{0m}$ and still succeed in avoiding any null-cone shadow on the earth Figure 3a may be used together with Eq (5 2) to find τ for a given antenna angle γ_0 and any given set of orbital parameters For example, if $\gamma_0 = 20°$ and $\alpha_1 = 21\ 15°$, the critical tilt

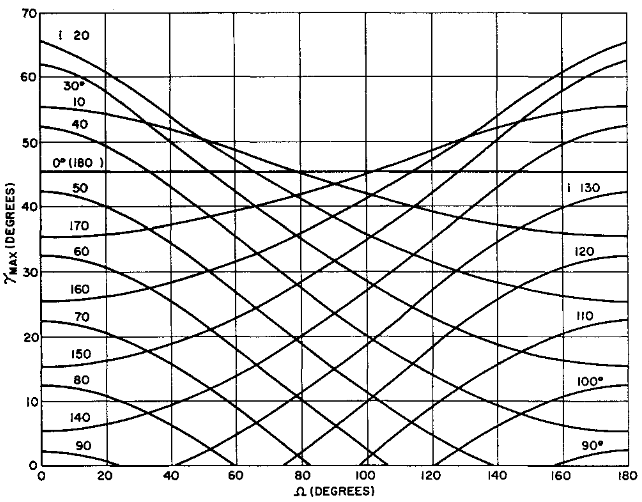

Fig 3b Maximum half angle of null cone for 6000-naut-mile orbit with minimum antenna elevation of $e_m = 7\frac{1}{2}°$

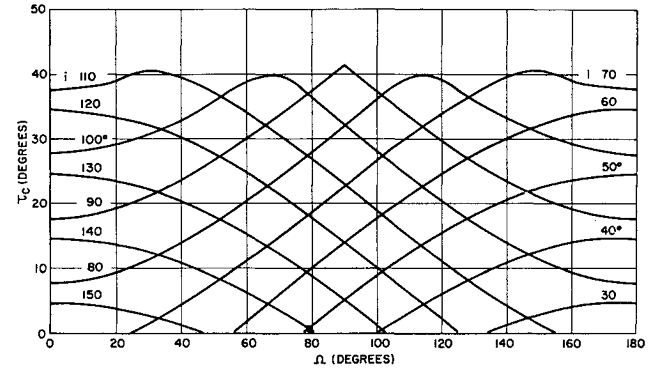

Fig 3c Critical tilt angle for null-cone half angle $\gamma_0 = 20°$ and 6000-naut-mile orbit with minimum antenna elevation $e_m = 7\frac{1}{2}°$

angle for various orbits is as shown in Fig 3c In using Eq (5 2), one must set $\tau = 0$ when $\beta_{0m} > \alpha_1 + \gamma_0$ since no shadow is present in such cases

If one is willing to accept some specified amount of null-cone shadow, but wishes to keep the tilt angle τ below some specified angle, it may prove desirable to tilt through some fraction of τ

For any given value of τ, the acute angle β between the spin axis OS and the radius vector OQ may be found from the right spherical triangle SQ_cQ shown in Fig 1 Upon noting that arc $Q_cQ = 90° - u'$, it follows that

$$\cos\beta = |\cos\beta_m \sin u'| = |\cos(\beta_{0m} + \tau) \sin u'| \qquad (5\ 3)$$

In general, the shadow pattern will be symmetrical about $u' = 90°$ and $u' = 0$, so that one need only explore in detail the range $0 < u' \leq 90°$

6 Calculation of Null-Cone Shadow

For any tilt angle τ, null-cone shadows will develop if $\beta < \beta_1 = \alpha_1 + \gamma$ [Eq (3 5)] In order to calculate the area covered by the null-cone shadow, it is desirable, first of all, to find the curve of intersection (CB in Fig 2a) of the null cone and the earth's surface To do this we note that a typical generator of the cone will pierce the earth's surface at a point I, as shown in Fig 4; this typical generator makes an angle σ with the satellite radius vector OQ and an angle γ with the spin axis The spin axis makes an acute angle β with the radius OQ as in previous discussions The piercing point I is fully defined by the spherical coordinates Φ and θ measured as shown in Fig 4 In order to express Φ and θ in terms of the single parameter σ it is convenient to construct the triangles OQD and JQD Point D represents the piercing point of the spin axis and the diametral plane (of the earth) which is perpendicular to OQ The point J represents the

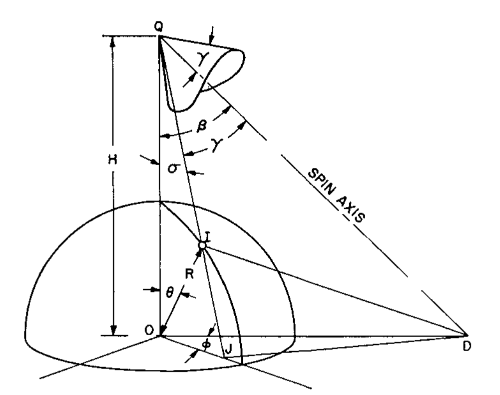

Fig 4 Piercing point of null-cone generator and earth's surface

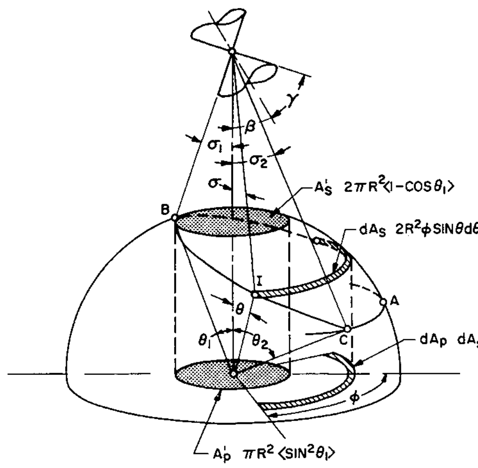

Fig 5 Area elements of null-cone shadow

piercing point of the cone generator and the same diametral plane The segment JD may be found from triangle ODJ in the form

$$(JD)^2 = (OD)^2 + (OJ)^2 - 2(OJ)(OD)\cos\Phi$$
$$= (H\tan\beta)^2 + (H\tan\sigma)^2 - 2H^2\tan\beta\tan\sigma\cos\Phi \quad (6\ 1)$$

Similarly, from triangle QJD,

$$(JD)^2 = (QJ)^2 + (QD)^2 - 2(QJ)(QD)\cos\gamma$$
$$= (H\sec\sigma)^2 + (H\sec\beta)^2 - 2H^2\sec\sigma\sec\beta\cos\gamma \quad (6\ 2)$$

Upon elimination of $(JD)^2$ from Eqs (6 1) and (6 2), one finds after some trigonometric simplification

$$\cos\Phi = (\cos\gamma - \cos\sigma\cos\beta)/(\sin\sigma\sin\beta) \quad (6\ 3)$$

Equation (6 3) predicts two values of Φ symmetrically dis-

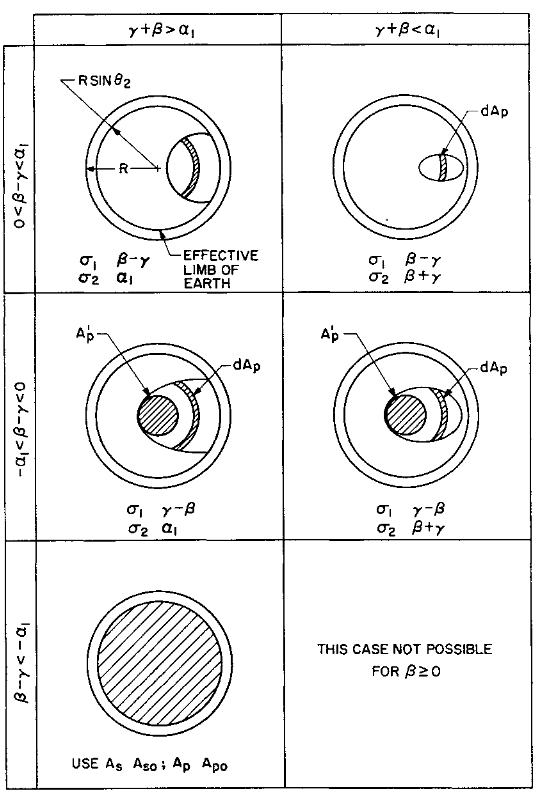

Fig 6 Limits of integration σ_1 and σ_2; summary of all possibilities for $\beta - \gamma \leq \alpha_1$

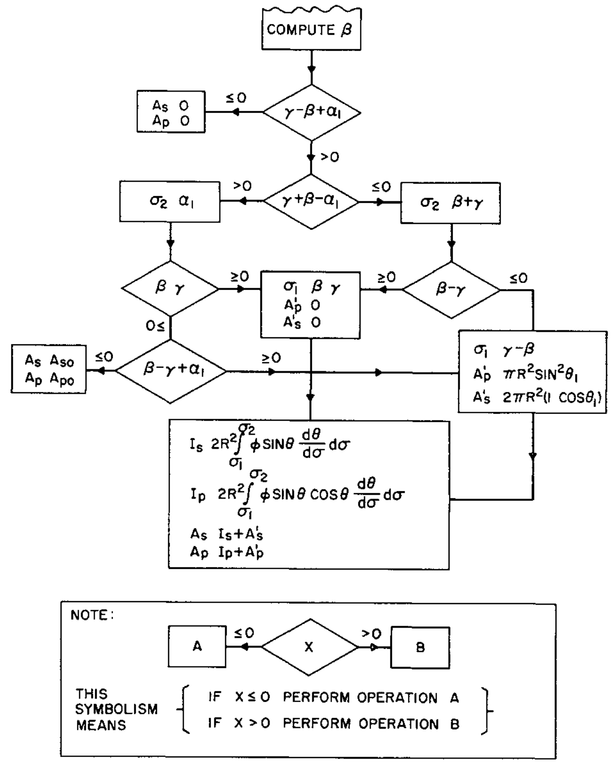

Fig 7 Flow chart of logic needed to choose limits of integration (σ_1, σ_2) and correct values of A_s' and A_p'

placed about $\Phi = 0$, as required For computational convenience, one may restrict Φ to the range $0 < \Phi < 180°$ and find the other half of the curve by reflection From triangle QOI, one finds $\sin(\theta + \sigma)/H = (\sin\sigma)/R$, where we are only concerned with the range $90° \leq \measuredangle QIO \leq 180°$; therefore,

$$\theta = \sin^{-1}[(H/R)\sin\sigma] - \sigma \qquad 0 < (\theta + \sigma) < 90° \quad (6\ 4)$$

Equations (6 3) and (6 4) give the coordinates of the locus of piercing point I as the parameter σ sweeps out all values in the range $\sigma_1 \leq \sigma \leq \sigma_2$, where the minimum and maximum values of σ are designated by σ_1 and σ_2, respectively

From Fig 5 it is apparent that, for a given value of σ, the surface area element dA bounded by two parallel circles and the sides of the null-cone shadow may be expressed as

$$dA = 2R^2\Phi(\sin\theta)d\theta \quad (6\ 5)$$

Hence, the sum of all such area elements is given by

$$I = \int dA = 2R^2 \int_{\sigma_1}^{\sigma_2} \Phi(\sin\theta)\frac{d\theta}{d\sigma}\,d\sigma \quad (6\ 6)$$

If the boundaries of the shadow BCA enclose the pole of the visible hemisphere [i e , if $\Phi = 180°$ when $\sigma = \sigma_1$, or $\Phi(\sigma_1) = \Phi_1 = 180°$] as shown in Fig 5, then the summation of all area elements dA described by Eq (6 5) will not include the polar cap of area A' shown in Fig 5 Note that if the shadow does not enclose the pole of the hemisphere (i e , if $\Phi = 0$ when $\sigma = \sigma_1$) the polar cap need not be included in the null-cone shadow Hence, the total surface area of the null-cone shadow A_s is given by

$$A = I + A' \quad (6\ 6a)$$

$$A = 2R^2 \int_{\sigma_1}^{\sigma_2} \Phi\sin\theta\left(\frac{d\theta}{d\sigma}\right)d\sigma + 2\beta R^2\langle 1 - \cos\theta_1\rangle \quad (6\ 6b)$$

where

$$\langle x\rangle = x \text{ if } \Phi = 180° \text{ when } \sigma = \sigma_1$$
$$= 0 \text{ if } \Phi = 0 \quad \text{when } \sigma = \sigma_1 \quad (6\ 7)$$

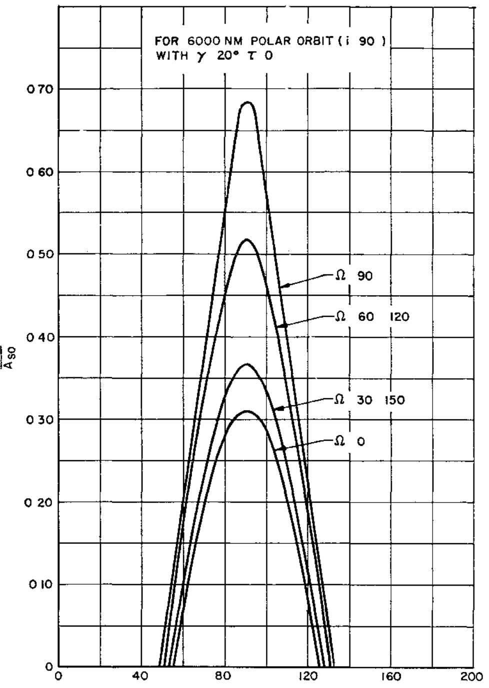

Fig 8a Fraction of surface area shadowed vs position in orbit; spin axis perpendicular to ecliptic plane

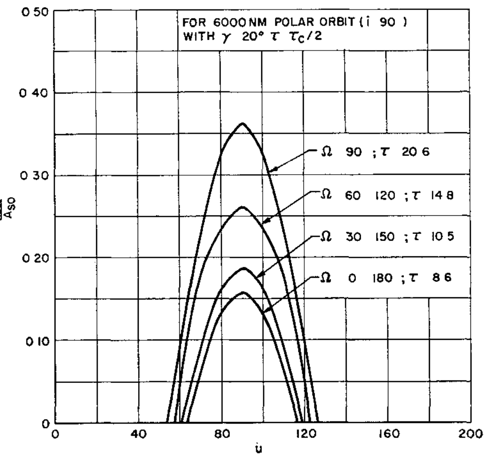

Fig 8b Fraction of surface area shadowed vs position in orbit when spin axis is tilted by angle $\tau = \tau_c/2$

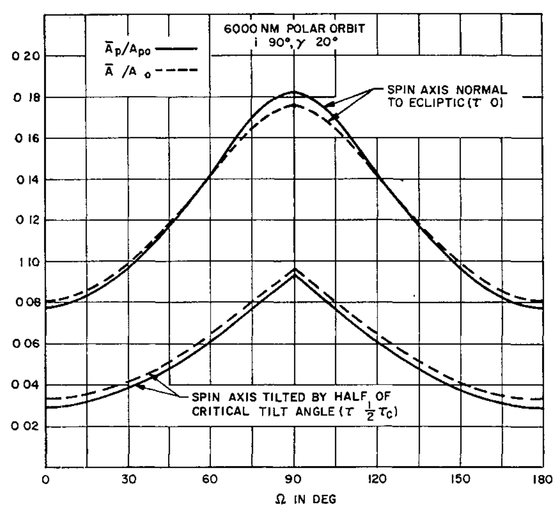

Fig 8c Average surface area $\bar{A}_s$ and projected area $\bar{A}_p$ vs longitude of node Ω

If use is made of the expression for A_{s0} given by Eq (3 1), and if the value of θ at $\sigma = \sigma_1$ is denoted by θ_1, one may nondimensionalize Eq (6 6) in the form

$$\frac{A_s}{A_{s0}} = \frac{\int_{\sigma_1}^{\sigma_2} \Phi \sin\theta \frac{d\theta}{d\sigma}\, d\sigma}{\pi(1 - \cos\theta_2)} + \frac{\langle 1 - \cos\theta_1 \rangle}{(1 - \cos\theta_2)} \qquad (6\ 8)$$

The integrand of Eq (6 8) is a well-defined function of the single variable σ by virtue of Eqs (6 3), (6 4), and the fact that Eq (6 4) may be differentiated to give

$$(d\theta/d\sigma) = [(H/R)(\cos\sigma)\cos(\sigma + \theta)] - 1 \qquad (6\ 9)$$

Thus, A_s/A_{s0} is easily found on a computer by means of Simpson's rule or some other method of numerical quadrature, once the limits of integration σ_1 and σ_2 are specified

In order to find the projected aiea A_p, it is only necessary to note that the aiea element dA_p is found by projecting dA_s onto the base plane by means of the simple relationship

$$dA_p = dA_s \cos\theta \qquad (6\ 10)$$

Upon summing up all area elements such as dA_p, one finds the projected area

$$I_p = \int dA_p = 2R^2 \int_{\sigma_1}^{\sigma_2} \Phi \sin\theta \cos\theta \frac{d\theta}{d\sigma}\, d\sigma \qquad (6\ 11)$$

The total projected area of the null-cone shadow A_p is found by adding the area A_p' (shown in Fig 5) when required, i e ,

$$A_p = I_p + A_p' \qquad (6\ 12)$$

Hence, one finds, after referring to Eq (3 2), that

$$\frac{A_p}{A_{p0}} = \frac{\int_{\sigma_1}^{\sigma_2} \Phi \sin\theta \cos\theta \frac{d\theta}{d\sigma}\, d\sigma}{\pi \sin^2\theta_2} + \frac{\langle \sin^2\theta_1 \rangle}{\sin^2\theta_2} \qquad (6\ 13)$$

In order to compute the integrals (6 6) or (6 11), it remains only to define specifically the limits of integration σ_1 and σ_2 If $\alpha_1 \leq \beta - \gamma$, one infers from inequality (3 6) that $A_p = A_s = 0$, so that it is only necessary to investigate the case $\beta - \gamma < \alpha_1$ If $\beta - \gamma \leq -\alpha_1$ (i e , if $\gamma - \beta \geq \alpha_1$), it may be seen from Fig 5 (or Fig 2a) that the cone generators at $\Phi = 0$ and $\Phi = 180°$ enclose the effective limb of the earth; hence $A_s = A_{s0}$ and $A_p = A_{p0}$ The only other possible values of β γ fall inside the range $-\alpha_1 < \beta - \gamma < \alpha_1$ The values of σ_1 and σ_2 inside this range depend upon whether or not the null-cone shadow is bounded by the limb of the effective earth disk; this, in turn, depends upon the value of $\alpha + \beta$ as shown in Fig 6 In Fig 6 are shown views of the earth looking down from the satellite When the null-cone shadow is bounded by the effective limb of the earth, $\gamma + \beta \geq \alpha_1$ (and $\sigma_2 = \alpha_1$); otherwise $\gamma + \beta \leq \alpha_1$ (and $\sigma_2 = \beta + \gamma$) When the null-cone shadow encloses the origin, in the views shown, γ exceeds β so that $\beta - \gamma < 0$ (and $\sigma_1 = \gamma - \beta$) When the shadow does not enclose the origin, β must exceed γ so that $0 \leq \beta - \gamma$ (and $\sigma_1 = \beta - \gamma$), as indicated in Fig 6

Numerical computations for surface area and projected area have been programmed on the IBM 7090 digital computer A logical program for choosing the correct limits of integration and correct values of A_p' and A_s' is outlined on the flow chart shown in Fig 7 Some typical results for satellites with a null cone having $\gamma_0 = 20°$ and polar orbits ($i = 90°$) are shown in Figs 8a, 8b, and 8c Figures 8a and 8b show the fraction of earth area shadowed (A_s/A_{s0}) when the spin axis is kept exactly perpendicular to the ecliptic ($\tau = 0$) and when the spin axis is tilted through half its critical angle ($\tau = \tau_c/2$), respectively Figure 8c shows how the average value of (A_s/A_{s0}) varies with the longitude of the node for various tilt angles (The symbol $\bar{A}_s/A_{s0}$ represents the value of A_s/A_{s0} averaged over a complete orbit) In all numerical work, the minimum antenna elevation e_m was assumed to be $7\frac{1}{2}°$, and the altitude was taken as 6000 naut miles ($H = 9440$ naut miles)

7 Effects of Orbital Precession

Because of the earth's oblateness, the longitude of the node Ω does not remain constant but decreases or increases (accordingly as the orbit is diıect or retrograde) according to the formula[7]

$$d\Omega/dt = \mp Jn \cos i/(H/R)^2$$

where $(d\Omega/dt)$ is the rate of nodal regression measured in the same units as n, the angulaı speed of the satellite in orbit, and $J = 1\ 624 \times 10^{-3}$; the upper sign is for direct orbits, and the lower sign is for retrograde orbits The effect of nodal motion may tend to either decrease or increase the null-cone shadow depending upon the instantaneous orbital elements If, for example, the satellite is at an inclination of 80°, Fig 3a shows that β_{0m} increases as Ω decreases for $0 < \Omega < 114°$, but β_{0m} increases with Ω for $114° < \Omega < 180°$ Hence, if the satellite is in a direct orbit, the decrease in Ω tends to decrease the shadow for $0 < \Omega \leq 114°$, but tends to increase the shadow for $114 \leq \Omega \leq 0$ For satellites with lifetimes of three to five years in near polar orbits, this effect is small

8 Conclusion

A general method has been established for finding the shape and area of the null-cone shadow on earth for arbitrary positions of the spin axis It has been shown that no null-cone shadow exists for a wide variety of nonpolar orbits, even though a finite null-cone angle is used (thereby providing some antenna gain) and the spin axis is kept perpendicular to the ecliptic plane Foı near polar orbits, some null-cone shadow is inescapable for antennas which provide gain (i e , nonvanishing null-cone angles) if the spin axis is kept exactly normal to the ecliptic plane However, if the spin axis is tilted off the normal to the ecliptic in a specified direction, the shadow may be reduced and even eliminated It has been shown that the amount of tilt is not unreasonable for many practical cases Generalized charts have been presented which allow one to pick the maximum null-cone angle (maximum gain antenna) that completely clears the eaıth of shadow or, wheıe this is not practical, how to find the critical tilt angle for any given antenna angle and any given elements of a circular orbit A computer program has been written which enables one to find the shape and size of the null-cone shadow throughout the course of any circular orbit for any given antenna angle and tilt angle

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